* Canonical Transformation Generating function :-

When a particle move on a circular motion Then at the surface of circle two coordinate (x, y) are needed to describe position of the particle And those coordinate does not presents in the cyclic coordinate But the value of those coordinate give the Lagrangian of the system with polar coordinate (r, θ)

from above it is clear that if a coordinate change into another coordinate then some coordinate behave as the cyclic coordinate

let q_j And p_j describe the position of the particle in Any system And it's corresponding hamiltonian is H

Now from hamiltonian variation principle the total variation in the system from t_1 to t_2 can be written as

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \qquad \left[\begin{array}{l} H = p_j \dot{q}_j - L \\ L = p_j \dot{q}_j - H \end{array}\right]$$

$$\delta \int_{t_1}^{t_2} (p_j \dot{q}_j - H)\,dt = 0 \quad \text{--- (1)}$$

Now the new coordinate of the system will be P_j And Q_j And it's corresponding hamiltonian is K.

Then it may be written as

$$\delta \int_{t_1}^{t_2} (P_j \dot{Q}_j - K)\, dt = 0 \quad \text{--- (2)}$$

Now let us consider a function F is know as generating function

$$\delta \int_{t_1}^{t_2} \frac{\partial F}{\partial t}\, dt = 0 \quad \text{--- (3)}$$

Subtraction eq 1 − (2+3)

$$\delta \int_{t_1}^{t_2} \left(p_j \dot{q}_j - H - P_j \dot{Q}_j + K - \frac{\partial F}{\partial t} \right) dt = 0$$

$$\delta \int_{t_1}^{t_2} \left(p_j \dot{q}_j - P_j \dot{Q}_j - \frac{\partial F}{\partial t} \right) dt = 0$$

Since H & K are the hamiltonian of old coordinate and new coordinate and hamiltonian is the sum of P.E and K.E and value of hamiltonian remains same for both the coordinate then above eq can be written as

$$\delta \int_{t_1}^{t_2} (p_j \dot{q}_j - P_j \dot{Q}_j)\, dt = \delta \int_{t_1}^{t_2} \frac{\partial F}{\partial t}\, dt$$

or

$$\boxed{p_j \dot{q}_j - P_j \dot{Q}_j = \frac{\partial F}{\partial t}}$$

This is the required expression for canonical transformation if any coordinate satisfy the above condition then they are canonical transform and function is known as generating function

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Imp.

* Poission bracket :- It is denoted by Rctuangular bracket [ ].
Let us Consider two Co-ordinate F & G whose Co-ordinate are $q_1 q_2 \ldots q_n$ And $p_1 p_2 \ldots p_n$ It means that F & G are the Function of position Co-ordinate And momentum Co-ordinate then the poission bracket may be written as

$$[F, G]_{q_j p_j}$$

$$\boxed{[F, G]_{q_j p_j} = \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}}$$

from above It is clear that the poission bracket for the Constant Term should be Equal to zero.

Propaty of poission bracket.

1. Poission bracket is Anti Symmetric in Nature

$$[F, G]_{q_j p_j} = -[G, F]_{q_j p_j}$$

Taking L.H.S of the above Eq

$$[F, G]_{q_j p_j}$$

$$\| \; [F, G]_{q_j p_j} = \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}$$

$$= -\left[\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}\right]$$

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$$\boxed{[F, G]_{q_j p_j} = -[G, F]_{q_j p_j}}$$

2. $[F, F]_{q_j p_j} = 0$

Taking L.H.S of the above Eq.

$$[F, F]_{q_j p_j}$$

$$[F, F]_{q_j p_j} = \cancel{\frac{\partial F}{\partial q_j}\frac{\partial F}{\partial p_j}} - \cancel{\frac{\partial F}{\partial p_j}\frac{\partial F}{\partial q_j}}$$

$$= 0$$

$$\boxed{[F, F]_{q_j p_j} = 0}$$

3. $[F_1 + F_2, G]_{q_j p_j} = [F_1, G]_{q_j p_j} + [F_2, G]_{q_j p_j}$

$$[F, G]_{q_j p_j} = \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}$$

Similarly

$$[F_1 + F_2, G]_{q_j p_j} = \frac{\partial (F_1 + F_2)}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial (F_1 + F_2)}{\partial p_j}\frac{\partial G}{\partial q_j}$$

$$= \left[\frac{\partial F_1}{\partial q_j} + \frac{\partial F_2}{\partial q_j}\right]\frac{\partial G}{\partial p_j} - \left[\frac{\partial F_1}{\partial p_j} + \frac{\partial F_2}{\partial p_j}\right]\frac{\partial G}{\partial q_j}$$

$$= \left[\frac{\partial F_1}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F_1}{\partial p_j}\frac{\partial G}{\partial q_j}\right] + \left[\frac{\partial F_2}{\partial q_j}\frac{\partial G}{\partial q_j} - \frac{\partial F_2}{\partial p_j}\frac{\partial G}{\partial q_j}\right]$$

$$= [F_1, G]_{q_j p_j} + [F_2, G]_{q_j p_j}$$

$$= \text{R.H.S}$$

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* With the help of poission brackets we can determine the canonical condition of any co-ordinate if the co-ordinate satisfy the following condition then they are said to be canonical transform.

$$[Q, P] = 1$$
$$[P, Q] = -1$$
$$[Q, Q] = [P, P] = 0$$

Q. Show that the transformation given by

$$p = \frac{p^2 + q^2}{2} \quad \text{— A}$$

$$Q = \tan^{-1}\left(\frac{q}{p}\right) \quad \text{— B}$$

is canonical

or $q = \sqrt{2p}\,\sin Q$
$p = \sqrt{2p}\,\cos Q$

Solⁿ:- by the condition of canonical transformation we know that

$$[Q, P]_{q_j p_j} = \frac{\partial Q}{\partial q_j}\frac{\partial P}{\partial p_j} - \frac{\partial Q}{\partial p_j}\frac{\partial P}{\partial q_j} = 1 \quad \text{— (1)}$$

Now we have to find the values

$\frac{\partial Q}{\partial q_j}, \frac{\partial P}{\partial p_j}$ And $\frac{\partial Q}{\partial p_j}, \frac{\partial P}{\partial q_j}$

So for find these values diff w.r.to $\frac{\partial Q}{\partial q_j}, \frac{\partial P}{\partial p_j}, \frac{\partial Q}{\partial p_j}, \frac{\partial P}{\partial q_j}$ of Eq (1) we get

A & B

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$$\frac{\partial Q}{\partial q_j} = \frac{p}{p^2+q^2}$$

$$\frac{\partial Q}{\partial p_j} = \frac{q}{p^2+q^2}$$

And $\frac{\partial P}{\partial p_j} = p$

$$\frac{\partial P}{\partial q_j} = q$$

Substitute these value in eq (1) we get

$$\frac{p^2}{p^2+q^2} + \frac{q^2}{p^2+q^2}$$

$$= 1$$

Hence the Transformation is Canonical.

Q. $P = \sqrt{2q}\, e^{-\alpha} \sin p$ — (A)

$Q = \sqrt{2q}\, e^{\alpha} \cos p$ — (B)

Sol^n: by the Condition of Canonical we have

$$[Q, P]_{q_j p_j} = \frac{\partial Q}{\partial q_j}\frac{\partial P}{\partial p_j} - \frac{\partial Q}{\partial p_j}\frac{\partial P}{\partial q_j} = 1 \quad \text{— (1)}$$

diff w.r.to $\frac{\partial Q}{\partial q}$, $\frac{\partial Q}{\partial p}$ And $\frac{\partial P}{\partial p}$, $\frac{\partial P}{\partial q}$ of eq A & B we get

$$\frac{\partial P}{\partial p} = \sqrt{2q}\, e^{-\alpha} \cos p$$

$$\frac{\partial P}{\partial q} = \frac{1}{\sqrt{2q}}\, e^{-\alpha} \sin p$$

$$\frac{\partial Q}{\partial q} = \frac{1}{\sqrt{2q}}\, e^{\alpha} \cos p$$

$$\frac{\partial Q}{\partial p} = -\sqrt{2q}\, e^{\alpha} \sin p$$

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Substituting these values in Eq (1) we get

$$\frac{1}{\sqrt{2}q} e^{\alpha} \cos p \times \sqrt{2}q\, e^{-\alpha} \cos p + \sqrt{2}q\, e^{\alpha} \sin p \times \frac{1}{\sqrt{2}q} e^{-\alpha} \sin p$$

$$\cos^2 p + \sin^2 p$$

$$= 1$$

Hence the Transformation is Canonical.

Paper

Q. for what values of m & n the Transformation Eq

$$Q = q^m \cos np$$

$$P = q^n \sin np$$

is Canonical.

Sol^n. given that

$$Q = q^m \cos np \quad \text{—(A)}$$

$$P = q^n \sin np \quad \text{—(B)}$$

Now we know that by the Condition of Canonical we have

$$[Q, P]_{q_j p_j} = \frac{\partial Q}{\partial q_j}\frac{\partial P}{\partial p_j} - \frac{\partial Q}{\partial p_j}\frac{\partial P}{\partial q_j} = 1 \quad \text{—(2)}$$

diff w.r.to

$\frac{\partial Q}{\partial q}, \frac{\partial Q}{\partial p}, \frac{\partial P}{\partial p}, \frac{\partial P}{\partial q}$ of Eq A & B

we get

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[redacted]

Z-Time

$$\frac{\partial Q}{\partial q} = mq^{m-1}\cos np$$

$$\frac{\partial Q}{\partial p} = -nq^{m}\sin np$$

$$\frac{\partial P}{\partial p} = nq^{m}\cos np$$

$$\frac{\partial P}{\partial q} = mq^{m-1}\sin np$$

Substituting these value in Eq (i) we get

$$mq^{m-1}\cos np \times nq^{m}\cos np + nq^{m}\sin np \times mq^{m-1}\sin np = 1$$

$$= mnq^{2m-1}\cos^2 np + mnq^{2m-1}\sin^2 np = 1$$

$$= mnq^{2m-1} \times 1 = 1$$

$$q^0 = mnq^{2m-1} = 1$$

Equating the coefficients of q

$$2m-1 = 0$$

$$m = 1/2$$

$$mnq^0 = 1$$

$$n = 2$$

Substituting the value of m & n in Eq A & B we get

$$Q = q^{1/2}\cos 2p$$

$$P = q^{1/2}\sin 2p.$$

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[redacted]

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